

Critical Path Analysis

BASIC CONCEPTS AND FORMULA

Basic Concepts

1. Framework of Pert/Cpm

The PERT and CPM models are extremely useful for the purpose of planning, scheduling and controlling the progress and completion of large and complex projects or for carrying out the analysis of these three managerial functions. A *network* is a graphical representation of a project, depicting the flow as well as the sequence of well-defined activities and events. Both CPM (Critical Path Method) and PERT (Programme Evaluation and Review Technique) are network techniques/ models.

2. Network

A network is, then, a graphical representation of a project plan, showing the inter-relationship of the various activities. Networks are also called *arrow diagrams* (see figure-6). When the results of time estimates and computations have been added to a network, it may be used as a project schedule.

3. Steps in PERT/CPM Model

PERT/CPM model building consists of following five steps:

1. Analyse and break down the project in terms of specific activities and/ or events.
2. Determine the interdependence and sequence of specific activities and prepare a network.
3. Assign estimates of time, cost or both to all the activities of the network.
4. Identify the longest or critical path through the network.
5. Monitor, evaluate and control the progress of the project by replanning, rescheduling and reassignment of resources.

4. Critical Path

The longest path is the critical path because it equals the minimum time required to complete the project. All other paths other than the critical path (i.e. non-critical or slack paths) offer flexibility in scheduling and transferring resources, because they

take less time to complete than the critical path.

5. Activity

An activity is a distinct operation or an element of a project which consumes time or resources and has a definable beginning and ending. Commonly used terms synonymous with "activity" are "task" and "job".



6. Conventions Adopted In Drawing Networks

There are two conventions normally adopted while drawing networks:

- (a) Time flows from left to right.
- (b) Head events always have a number higher than that of the tail events.

7. Graphical Representation of Events and Activities

Events are represented by numbers within circles. Activities are represented by arrows; the arrow-heads represent the completion of the activities. The length and orientation of the arrow are of no significance.

8. Fundamental Properties Governing the Representation of Events and Activities

The representation of events and activities is governed by one simple *dependency rule* which requires that an activity which depends upon another activity is shown to emerge from the head event of the activity upon which it depends and that only dependent activities are drawn in this way. An event cannot occur until all activities leading to it are complete. No activity can start until its tail event is reached.

9. Logical Sequencing and Connection of Activities

A project entails several activities. The arrows are arranged to show the plan of logical sequence in which the activities of the project are to be accomplished. The sequence is ascertained for each activity by three queries viz:

- (i) Which activity or activities must be completed before the start of a particular activity?
- (ii) Which activity or activities should follow this?
- (iii) Which activities can be accomplished simultaneously?

10. Errors in logical sequencing

Two types of errors in logic may arise while drawing a network, particularly when it is a complicated one. These are known as *looping* and *dangling*.

11. Dummy activity

It is a hypothetical activity which consumes no resource and time. It is represented by dotted lines and is inserted in the network to clarify activity pattern under the following situations:

- (i) It is created to make activities with common starting and finishing events distinguishable.
- (ii) to identify and maintain the proper precedence relationship between activities that are not connected by events.
- (iii) to bring all "loose ends" to a single initial and a single terminal event in each network using dummies, if necessary.

Question 1

Explain the following in the context of a network:

- (i) Critical path
- (ii) Dummy activity.

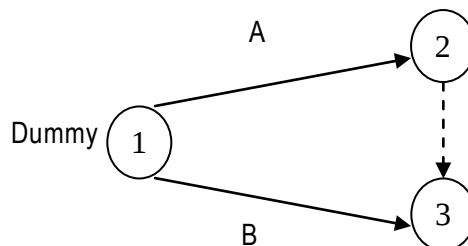
Answer

(i) Critical Path:

Critical Path is a chain of activities that begin with the starting event and ends with ending event of a particular project. It is that path that runs through a network with the maximum length of time or it indicates the maximum possible time required for completion of a project. Critical path indicates the minimum time that will be required to complete a project. It is determined after identifying critical events. Critical path goes through critical events.

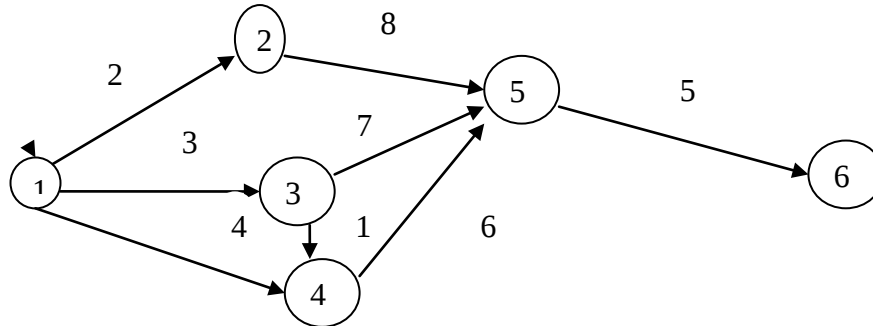
(ii) Dummy Activities:

Dummy Activity is that activity which does not consume time or resources. It is used when two or more activities have same initial and terminal events. As a result of using dummy activities, other activities can be identified by unique end events. These are usually shown by arrows with dashed lines.



Question 2

The following network gives the duration in days for each activity:



- (i) You are required to list the critical paths.
- (ii) Given that each activity can be crashed by a maximum of one day, choose to crash any four activities so that the project duration is reduced by 2 days.

Answer

Critical Paths:

All are critical paths:

- | | | |
|-------|-------------------|----------------------|
| (i) | 1 – 2 – 5 – 6 | $2 + 8 + 5 = 15$ |
| (ii) | 1 – 3 – 5 – 6 | $3 + 7 + 5 = 15$ |
| (iii) | 1 – 4 – 5 – 6 | $4 + 6 + 5 = 15$ |
| (iv) | 1 – 3 – 4 – 5 – 6 | $3 + 1 + 6 + 5 = 15$ |

- (i) Choose 5 – 6, common path;

Crash by 1 day

- (ii) Choose: 1 – 2, 1 – 3, 1 – 4

Or

- (iii) Choose: 1 – 2, 3 – 5, 4 – 5

Or

- (iv) Choose: 2 – 5, 3 – 5, 4 – 5

Or

- (v) Choose: 1 – 3, 1 – 4, 2 – 5

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Question 3

A company is launching a new product and has made estimates of the time for the various activities associated with the launch as follows:

Activity	Predecessor	Times (Days)		
		Optimistic	Most likely	Pessimistic
A	None	1	3	5
B	None	3	4	5
C	A, B	1	3	11
D	B	3	3	9
E	A	1	2	3
F	C	2	5	14
G	E, F	2	3	4
H	D, F	2	2	2
I	G, H	10	10	10

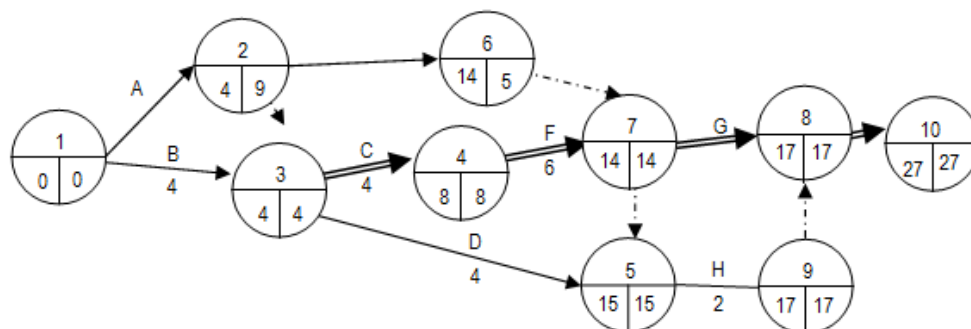
Required:

- Draw the network diagram.
- Calculate the expected time and variance of each activity.
- Find out the expected length of critical path and its standard deviation.
- Find the probability that the launching will be completed in 27 days.
- Find the duration, which has 95% probability of completion.

Answer

(i) Network Diagram

Critical Path A → B → C → F → G → Z



(ii) Calculation of Expected Time, Standard Deviation and Variance of Activities

Activity	Expected Time $t_e = \frac{t_o + 4t_m + t_p}{6}$	Standard Deviation $S = \frac{t_p - t_o}{6}$	Variance σ^2
A (1-2)	$\frac{1+12+5}{6} = 3$	$\frac{5-1}{6} = 0.67$	0.44
B (1-3)	$\frac{3+16+5}{6} = 4$	$\frac{5-3}{6} = 0.33$	0.11
C (3-4)	$\frac{1+12+11}{6} = 4$	$\frac{11-1}{6} = 1.67$	2.78
D (3-5)	$\frac{3+12+9}{6} = 4$	$\frac{9-3}{6} = 1.00$	1.00
E (2-6)	$\frac{1+8+3}{6} = 2$	$\frac{3-1}{6} = 0.33$	0.11
F (4-7)	$\frac{2+20+14}{6} = 6$	$\frac{14-2}{6} = 2.00$	4.00
G (6-8)	$\frac{2+12+4}{6} = 3$	$\frac{4-2}{6} = 0.33$	0.11
H(5-9)	$\frac{2+8+2}{6} = 2$	$\frac{2-2}{6} = 0$	0
I (8-10)	$\frac{10+40+10}{6} = 10$	$\frac{10-10}{6} = 0$	0

(iii) S.D. of Critical Path

$$= \sqrt{\text{Total of variance of Critical Activities}}$$

$$= \sqrt{0.11+2.78+4+0.11+0}$$

$$= \sqrt{7} = 2.645$$

(iv) Probabilities of completion of job in 27 days.

$$X = 27 \text{ Days}$$

$$Z = \frac{27 - 27}{2.645} = 0$$

For $Z = 0$ the probability is 0.5 from the table of area under normal curve or 50%.

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- (v) For 95% of area the corresponding Z value is 1.64 (from the table).

Therefore,

$$1.64 = \frac{X - 27}{2.645}$$

$$X = 27 + 4.33 = 31.33 \text{ Days}$$

Question 4

Consider the schedule of activities and related information as given below, for the construction of a Plant:

Activity	Expected Time (Months)	Variance	Expected Cost (Millions of Rs.)
1-2	4	1	5
2-3	2	1	3
3-6	3	1	4
2-4	6	2	9
1-5	2	1	2
5-6	5	1	12
4-6	9	5	20
5-7	7	8	7
7-8	10	16	14
6-8	1	1	4

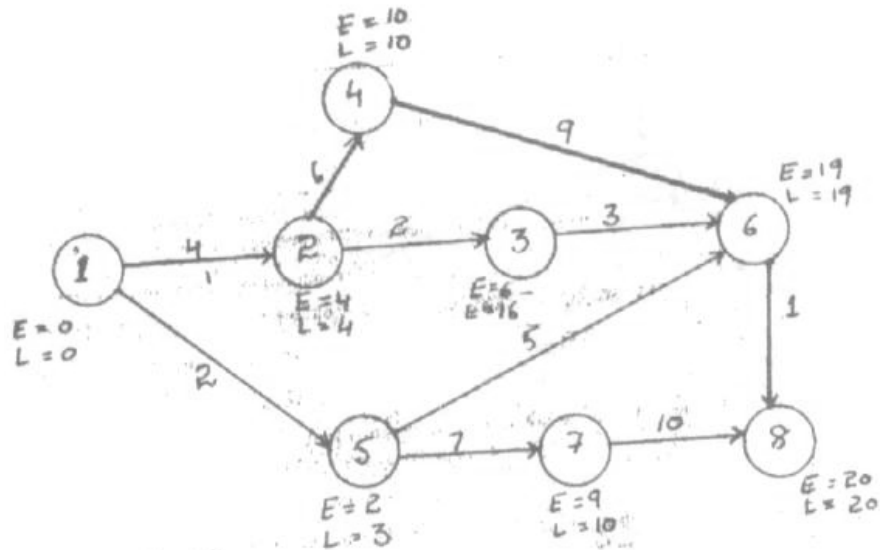
Assuming that the cost and time required for one activity is independent of the time and cost of any other activity and variations are expected to follow normal distribution.

Draw a network based on the above data and calculate:

- (i) Critical path
- (ii) Expected cost of construction of the plant
- (iii) Expected time required to build the plant
- (iv) The standard deviation of the expected time.

Answer

The required network is drawn below:



- (i) From the above network, it can be noted that the critical path is 1 – 2 – 4 – 6 – 8.
- (ii) Expected cost of construction of the plant = (5 + 3 + 4 + 9 + 2 + 12 + 20 + 7 + 14 + 4) millions of Rs. = Rs.80 million
- (iii) Expected time required to build the plant = 4 + 6 + 9 + 1 = 20 months.
- (iv) It is given that the time required for one activity is independent of the time and cost of any other activity and variations are expected to follow normal distribution, the S.D.

Hence, the variance of the expected time is determined by summing the variance of critical activities and is = 1 + 2 + 5 + 1 = 9.

Standard Deviation of the expected time = $\sqrt{9}$ = 3 months.

Question 5

A product comprised of 10 activities whose normal time and cost are given as follows:

Activity	Normal Time (days)	Normal cost
1-2	3	50
2-3	3	5
2-4	7	70
2-5	9	120

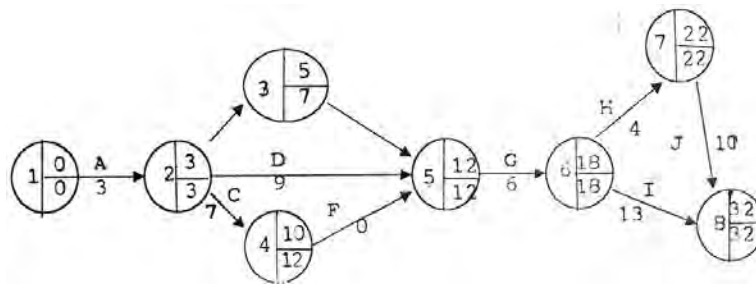
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3-5	5	42
4-5	0	0
5-6	6	54
6-7	4	67
6-8	13	130
7-8	10	166

Indirect cost Rs. 9 per day.

- Draw the network and identify the critical path.
- What are the project duration and associated cost ?
- Find out the total float associated with each activity.

Answer



Critical path A D G H J

1-----2-----5-----6-----7-----8

- A D G H J is the critical path having normal project duration
 $3 + 9 + 6 + 4 + 10 = 32$ days

Normal project cost:- Direct cost

= Rs. 704

Indirect cost (32' 9)

= 288

992

- Calculation of total float

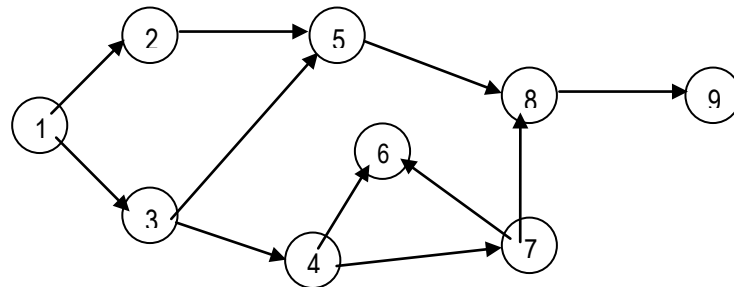
Activity	Nt(days)	E_F	L_F	Float ($L_F - E_F$)
1-2	3	3	3	0
2-3	3	6	7	1
2-4	7	10	12	2

2-5	9	12	12	0
3-5	5	11	12	1
4-5	0	10	12	2
5-6	6	18	18	0
6-7	4	22	22	0
6-8	13	31	32	1
7-8	10	32	32	0

Question 6

A network is given below:

- (i) Name the paths and give their total duration.
(ii) Give three different ways of reducing the project above duration by four days.

**Answer**

- (i) Assuming that the duration of activity 3 – 5 is 4 weeks.

The various critical paths are:

1-2-5-8-9	15 weeks
1-3-4-7-8-9	15 weeks
1-3-4-6-7-8-9	15 weeks
1-3-5-8-9	15 weeks

- (ii) **Note:** Since the duration for activity 3-5 is not specified it is open for you to assume the duration. Depending upon the duration assume **three** possibilities emerge.
1. If the duration assumed is more than 4 weeks then that path (13, 35, 58, 89) alone will be critical. In that case you can choose any of the activity in the critical path.
 2. If the duration assumed is exactly 4 weeks then it will be one of the 4 critical paths and the various possibilities are given below.

- Reduce in the following ways, the project duration is. Since all the paths are critical, reduction is possible by combining activities. The activities can be independent, common to few paths and common to all the paths. The various categories are as follows:

- (Any three of the above combination.)**

A company had planned its operations as follows:

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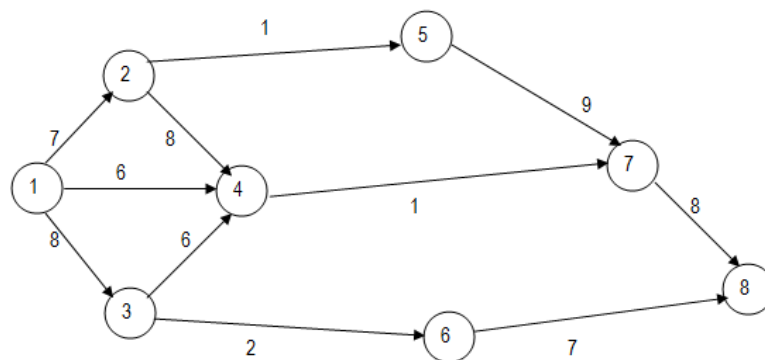
1-4	6
2-5	16
4-7	19
3-6	24
5-7	9
6-8	7
7-8	8

- (i) Draw the network and find the critical paths.
- (ii) After 15 days of working, the following progress is noted:
- Activities 1-2, 1-3 and 1-4 completed as per original schedule.
 - Activity 2-4 is in progress and will be completed in 4 more days.
 - Activity 3-6 is in progress and will need 17 more days to complete.
 - The staff at activity 3-6 are specialised. They are directed to complete 3-6 and undertake an activity 6-7, which will require 7 days. This rearrangement arose due to a modification in a specialisation.
 - Activity 6-8 will be completed in 4 days instead of the originally planned 7 days.
 - There is no change in the other activities.

Update the network diagram after 15 days of start of work based on the assumption given above. Indicate the revised critical paths alongwith their duration.

Answer

(i)



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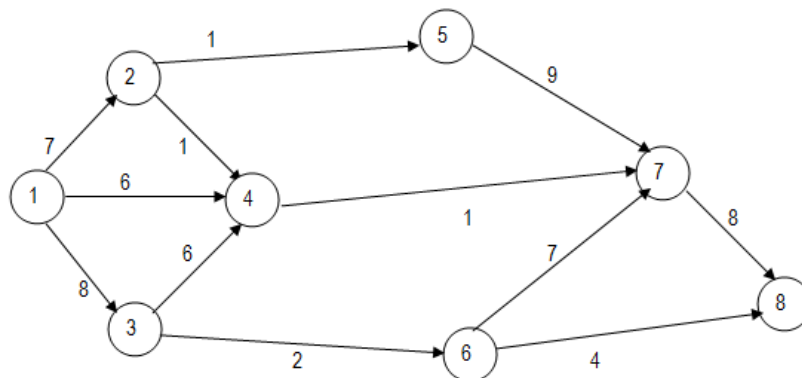
Paths	Duration
1 – 2 – 5 – 7 – 8	$7 + 16 + 9 + 8 = 40$
1 – 2 – 4 – 7 – 8	$7 + 8 + 19 + 8 = 42$
1 – 4 – 7 – 8	$6 + 19 + 8 = 33$
1 – 3 – 4 – 7 – 8	$8 + 6 + 19 + 8 = 41$
1 – 3 – 6 – 8	$8 + 24 + 7 = 39$

Critical path 1 – 2 – 4 – 7 – 8 = 42 days.

Revised Duration of activities 2 – 4 and 3 – 6 after 15 days for update.

Activity	Preceding Activity	Date of completion	Revised Duration
2 – 4	1 – 2	$15 + 4 = 19$ days	$19 - 7 = 12$ days
3 – 6	1 – 3	$15 + 17 = 32$ days	$32 - 8 = 24$ days
6 – 7 (new activity)	3 – 6		7 days
6 – 8	3 – 6		4 days

(ii)



Paths	Duration
1 – 2 – 5 – 7 – 8	$7 + 16 + 9 + 8 = 40$
1 – 2 – 4 – 7 – 8	$7 + 12 + 19 + 8 = 46$
1 – 4 – 7 – 8	$6 + 19 + 8 = 33$
1 – 3 – 4 – 7 – 8	$8 + 6 + 19 + 8 = 41$
1 – 3 – 6 – 7 – 8	$8 + 24 + 7 + 8 = 47$
1 – 3 – 6 – 8	$8 + 24 + 4 = 36$

Critical path = 1 – 3 – 6 – 7 – 8 = 47 days.

Question 8

The following table gives the activities in a construction project and the time duration of each activity:

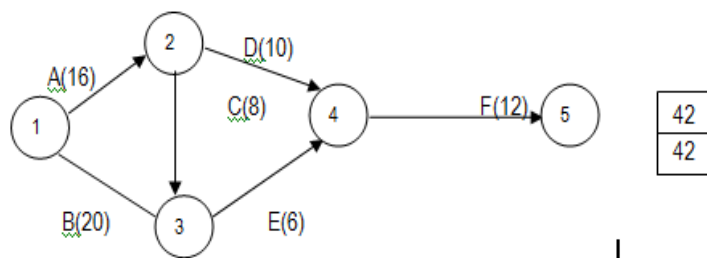
Activity	Preceding activity	Normal Time (Days)
A	-	16
B	-	20
C	A	8
D	A	10
E	B, C	6
F	D, E	12

Required:

- Draw the activity network of the project.
- Find critical path.
- Find the total float and free-float for each activity.

Answer

(i)



$$A \text{ ® } D \text{ ® } F = 16 + 10 + 12 = 38$$

$$B \text{ ® } E \text{ ® } F = 20 + 6 + 12 = 38$$

- (ii) $A - C - E - F = 16 + 8 + 6 + 12 = 42$ Critical path

- (iii) Total float and free float for each activity

Activity	Normal time (Days)	Earliest start	Time finish	Latest start	Time finish	Float total	Free
A	16	0	16	0	16	0	0
B	20	0	20	4	24	4	4
C	8	16	24	16	24	0	0

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D	10	16	26	20	30	4	4
E	6	24	30	24	30	0	0
F	12	30	42	30	42	0	0

Question 9

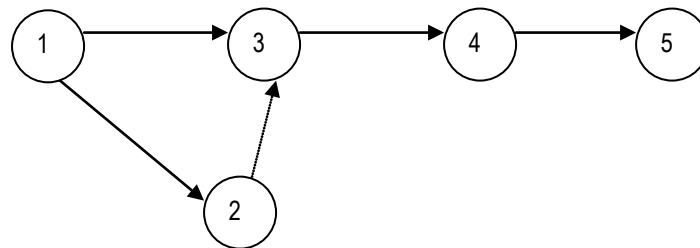
What do you mean by a dummy activity? Why is it used in networking?

Answer

Dummy activity is a hypothetical activity which consumes no resource or time. It is represented by dotted lines and is inserted in the network to clarify an activity pattern under the following situations.

- (i) To make activities with common starting and finishing events distinguishable.
- (ii) To identify and maintain the proper precedence relationship between activities that are not connected by events.
- (iii) To bring all "loose ends" to a single initial and single terminal event.

e.g.



Dummy (2) – (3) is used to convey that can start only after events numbered (2) and (3) are over:

Question 10

List the 5 steps involved in the methodology of critical path analysis.

Answer

Working Methodology of PERT:

The working methodology of PERT which includes both CPM and PERT, consists of following five steps:

1. Analyze and break down the project in terms of specific activities and/or events.
2. Determine the interdependence and sequence of specific activities and prepare a network.

3. Assign estimates of time, cost or both to all the activities of the network.
4. Identify the longest or critical path through the network.
5. Monitor, evaluate and control the progress of the project by re-planning, rescheduling and reassignment of resources.

Question 11

A small project is composed of seven activities, whose time estimates are listed below. Activities are identified by their beginning (i) and ending (j) note numbers:

Activity	Estimated durations (in days)		
(i-j)	Optimistic	Most likely	Pessimistic
1-2	2	2	14
1-3	2	8	14
1-4	4	4	16
2-5	2	2	2
3-5	4	10	28
4-6	4	10	16
5-6	6	12	30

- (a) Draw the project network.
- (b) Find the expected duration and variance for each activity. What is the expected project length?

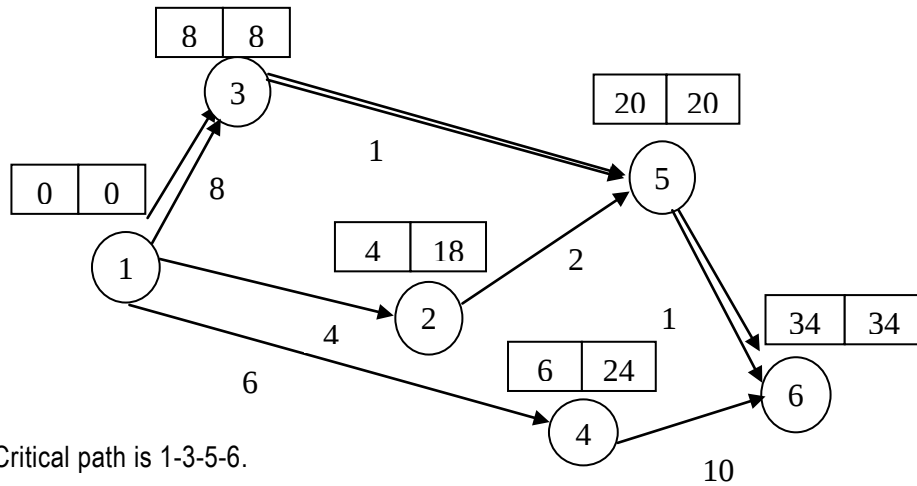
Given :	Z	0.50	0.67	1.00	1.33	2.00
	P	0.3085	0.2514	0.1587	0.0918	0.0228

Answer

Activity Estimated Durations :

$$\text{Activity estimated durations (days)} t_e = \frac{(a + 4m + b)}{6} \text{ Duration}$$

Activity	a	m	b	t_e	$s = [(b - a)/6]^2$
1-2	2	2	14	4	4
1-3	2	8	14	8	4
1-4	4	4	16	6	4
2-5	2	2	2	2	0
3-5	4	10	28	12	16
4-6	4	10	16	10	4
5-6	6	12	30	14	16



EXERCISE

Question 1

The time schedule for different activities of a project is given below:

Activity (i – j)	Time in days
1-2	8
1-3	10
1-4	8
2-3	10
2-6	16
3-5	17
4-5	18
4-6	14
5-6	9

Construct the PERT network and compute.

- Critical path and its duration.
- Total and free float for each activity.

Answer

The critical path is given by 1 – 2 – 3 – 5 – 6. The path represents the minimum possible time to complete the project.

The project duration = 8 + 10 + 17 + 9 = 44 days.

Question 2

A project has the following time schedule:

Activity	Time in weeks	Activity	Time in weeks
1-2	4	5-7	8
1-3	1	6-8	1
2-4	1	7-8	2
3-4	1	8-9	1
3-5	6	8-10	8
4-9	5	9-10	7
5-6	4		

Construct a PERT network and compute:

- (i) T_E and T_L for each event;
- (ii) Float for each activity; and
- (iii) Critical path and its duration.

Answer

Critical path is given by all those activities which have zero floats. Along the zero float activities, there are two such critical paths:

- (i) $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10$
- (ii) $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 10$

The project duration is 25 weeks.

Question 3

Given the following information:

Activity:	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7	
Duration:	2	8	10	6	3	3	7	5	2	8	(in days)

- (i) Draw the arrow diagram.
- (ii) Identify critical path and find the total project duration.
- (iii) Determine total, free and independent floats.

Answer

The critical path is - $0 \rightarrow 1 \rightarrow 3 \rightarrow 6 \rightarrow 7$

Total project duration = 27 days.